

ON THE NUMERICAL STUDY OF THE COMPLEXITY AND FRACTAL DIMENSION OF CMB ANISOTROPIES

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Abstract

We consider the problem of numerical computation of the Kolmogorov complexity and the fractal dimension of the anisotropy spots of Cosmic Microwave Background (CMB) radiation. Namely, we describe an algorithm of estimation of the complexity of spots given by certain pixel configuration on a grid and represent the results of computations for a series of structures of different complexity. Thus, we demonstrate the calculability of such an abstract descriptor as the Kolmogorov complexity for CMB digitized maps. The correlation of complexity of the anisotropy spots with their fractal dimension is revealed as well. This technique can be especially important while analyzing the data of the forthcoming space experiments.

1 Introduction

The study of the properties of Cosmic Microwave Background (CMB) radiation is known to be one of most profound means for revealing the early evolution of the Universe. Among such CMB characteristics as the amplitude of anisotropy, angular autocorrelation function, etc. the properties of CMB sky maps are also known to carry essential cosmological information [1][2] [3]. Particularly direct information can be obtained on the density parameter Ω and the primordial fluctuation spectrum, which themselves are important, though not always decisive, for the theoretical scenarios on the evolution of the early Universe. For example, the inflationary cosmological models initially developed to explain the flatness of the Universe among its other properties, are shown to predict also low density Universe with $\Omega < 1$ (for recent discussion of these cosmological aspects see [4]).

The CMB properties in negatively curved spaces contain differences as compared with flat or positively curved ones. Namely, the motion of photon beams in negatively curved homogeneous and isotropic spaces results in an effect of geodesic mixing [5]. In the case of Friedmannian Universe with $k=-1$, the following observable consequences of the effect of geodesic mixing have been predicted [5][6]: (1) damping of anisotropy after the last scattering epoch; (2) flattening of autocorrelation function; (3) distortion of anisotropy spots.

Since the effect is absent at $k = 0, +1$ Universe, by means of the analysis of these properties, in particular, of the sky maps, it is possible to obtain information on the ge-

ometry of the Universe. The predicted distortion, in the simplest case, can be attributed to elongation of anisotropy spots, directly depending on the curvature of the Universe, and hence, on the density parameter Ω , as well as on the redshift of the last scattering epoch, i.e. the distance covered by the photons while moving via geodesics. Indication for such elongation was found while studying the COBE 4 year data [7]. It was mentioned [6], however, that more precise property arising in the negatively curved spaces is the complexity of anisotropy spots, and the Kolmogorov complexity [9][10] being as possible descriptor of that property. Namely an expression can be derived [8] relating the complexity of CMB anisotropies on the curvature of the Universe.

In the present paper we describe a way of numerical treatment of the complexity of the spots, namely the algorithms and the results of calculations of the Kolmogorov complexity K and the Hausdorff (fractal) dimension d of the spots. We observe the correlated growth of the complexity K and d with the increase of the complexity of the geometrical shape of the spots, starting from the simplest case - the circle.

Together with the previous results on the rate of exponential mixing of geodesics determined by the Kolmogorov-Sinai (KS) entropy, which itself is determined by the diameter of the Universe, this provides a new informative way of the analysis of the sky map data. This is especially important given the forthcoming high precision CMB observation space programs - Planck Surveyor (ESA) and MAP (NASA).

We start from the brief account of effect of geodesic mixing, the concepts of Kolmogorov complexity and the Hausdorff dimension.

2 Geodesic mixing

The geodesics on spaces (locally if the space is non-compact) with negative curvature in all two dimensional directions are known to possess properties of Anosov systems, including an exponential instability and positive KS-entropy. Time correlation functions reflect the basic properties of dynamical systems and therefore the knowledge of their behavior is needed though not always a simple problem. As it was proved by Pollicot [11] for $\dim=3$ manifold M with constant negative curvature the time correlation function of the geodesic flow $\{f^\lambda\}$ on the unit tangent bundle SM of M is decreasing by exponential law for all smooth functions A, B

$$|b_{A,B}(\lambda)| \leq c \cdot |b_{A,B}(0)| \cdot e^{-h\lambda}, \quad (1)$$

where $c > 0$, h is the KS-entropy of the geodesic flow; for discussion of certain physical aspects of this property of dynamical systems see the monograph[6].

To reveal the properties of the free motion of photons in (3+1) Friedmann-Robertson-Walker space the projection of its geodesics into Riemannian 3-space has to be performed, i.e. by corresponding the curve $c(\lambda) = x(\lambda)$ to the curve in the former space: $\gamma(\lambda) = (x(\lambda), t(\lambda))$ [12]. The KS-entropy in the exponential index in (1) can be easily estimated for the matter dominated post-scattering Universe [5], so that

$$e^{h\lambda} = (1+z)^2 \left[\frac{1 + \sqrt{1-\Omega}}{\sqrt{1+z\Omega} + \sqrt{1-\Omega}} \right]^4. \quad (2)$$

i.e. depends on the density parameter Ω , the redshift of the last scattering epoch z . The exponential instability and hence the mixing over all phase space coordinates, as basic property of Anosov systems [13], results in the complex structures on the CMB sky maps:

the loss of the initial information by mixing leads to arising of complexity of images. Thus the complexity of the anisotropies we will estimate below has a dynamical nature, arising due to the properties of photon beam motion in space with specific geometry.

3 Complexity and Random Sequences

Before turning to the algorithm of computation of the complexity in our problem, we briefly represent the key definitions necessary to describe this universal concept. Indeed, during the recent decades the concept of complexity has become a key one in a broad area of fundamental problems - from the algorithmic information theory [10], up to the second law of thermodynamics and basics of statistical mechanics [17].

Already in 1965 Kolmogorov [9] introduced the concept of complexity, defined as a property of an object represented in a binary form; similar ideas almost simultaneously were developed by Solomonoff, Chaitin and by many authors later on (see [10][15][16]).

To define the complexity we need the following concepts:

Object. This is a general representation of an object, since every such sequence can be considered as a binary representation of an integer.

Computer. The latter is performing a set of 'deterministic' ¹ operations (addition, multiplication, division and other operations which can be performed by usual computers). A computer is considered 'universal' if for any computer C there exists a constant S_C which can be added to any program p , so that $S_C p$ should execute the same operation on computer U as the program p on computer C .

Algorithm. An algorithm for a computer is a set of instructions defining which operations have to be executed by the computer and when. Since the computer must halt, therefore, any program cannot be a prefix ² for some other program: The set of accessible programs should be *prefix-free*.

Complexity. The complexity $K_U(x)$ of the sequence x by a universal computer U is defined as the length in bits of the smallest algorithm p by which the computer U starting with some *initial fixed state* calculates the object x as its *only output*, and *halts*. Thus the sequence can be called complex if its complexity is comparable with its length. In this definition should be noted that the time of calculation is not important, and hence it can be chosen arbitrary.

Random sequences. The complexity is closely related with another basic concept - the random sequences. The most general definition by Martin-Löf [14][15] is formalizing the idea of Kolmogorov that random sequences have very small number of rules comparing to its length; the rule is defined as an algorithmically testable and rare property of a sequence. Indeed, the properties of complexity and randomness are not totally the same. But it is not surprising that these properties are closely related for typical sequences [15]. Therefore in our problem, in principle, the estimation of the randomness of the data string (digitized figure) has to correlate with the estimation of the complexity.

In certain trivial cases low-complexity objects can be distinguished easily, for example, $(0, \dots, 0)$ or $(1, \dots, 1)$. In some other cases, the object could seem to have a complex binary representation, such as π , though actually they are also of low-complexity. The estimation of complexity is simple, for example, for any integer, etc. In general case,

¹Here we define 'a computer' as a machine which can perform only deterministic operations. The classical results of Shannon et al (see [15]) show that for problems with unique solution probabilistic computers are not better: using *the random rules* only the time of computation can be decreased.

²a word a is called prefix for a word b if $b = ac$ with some other word c

however, the situation is much less simple. Moreover, it is proved that there is no a short algorithm to decide whether a given complex-looking sequence is really complex [10],[15].

Fortunately, though in general the shortest program cannot be reached, i.e. the exact complexity cannot be calculated, in certain problems the obtained results cannot be too far from that value.

If the length of a sequence x is N then the obvious upper limit can be established ³

$$K_U(x) < N. \quad (3)$$

Generally speaking what one can say about the complexity of typical string of a length N ? Let us estimate the fraction of such sequences (among all N -bit sequences) for which

$$K_U(x) < N - m.$$

This means that there exists a program of length $N - m$ which computes x . The total number of such programs of such a length cannot be larger than 2^{N-m+1} ; this is the upper limit without taking into account the prefix-free condition. Thus, we have the following upper limit

$$(2^{N-m+1} - 1)/2^N \approx 2^{-m+1}.$$

This value is small if m is sufficiently large. Thus a more general relation than (3) can be established

$$K_U(x) \approx c(x) N, c(x) \approx 1 \quad (4)$$

Thus, the calculation of relative complexity of an object and of a perturbed object via given computer and developed code (though the latter cannot be proved to be the shortest possible), has to reflect the complexity introduced by the perturbation. Since in our problem the complexity is a result of propagation of photons after the last scattering surface (if $k=-1$), one can thus 'measure the perturbation' caused by the curvature of the space as it was performed while measuring the elongation of the CMB anisotropy spots in [7].

4 Complexity: the Algorithm of Numerical Analysis

To develop the algorithm of estimation of complexity one should clearly describe in which manner the objects, namely the anisotropy spots, are defined. The COBE-DMR CMB sky maps have the following structure [3]. They represent a $M \times N$ grid with pixels determined by the beam angle of the observational device; more precisely the pixel's size defines the scale within which the temperature is smoothed, so that each pixel is assigned by certain value of temperature (number). For example, COBE's grid had 6144 pixels of about 2.9° size each, though they not uniformly contain the information on CMB photons. By 'anisotropy spots' we understand the sets of pixels at a given temperature threshold [18].

Our problem is to estimate the complexity of the anisotropy spots, i.e. of various configurations of pixels on the given grid: the size of the grid, and both the size and the number of pixels are crucial for the result.

We proceed as follows. Each row of the grid is considered as an integer of M digits in binary representation, '0' corresponding to the pixels not belonging to the spot, '1' - those of the spot. Considering all N rows of the grid in one sequence (the second row

³It should be noted that if x is the binary representation of some integer N_0 , then $N \approx \log_2 N_0$.

Table 1:

first 2 bits	next bits
0 1	1
1 0	2
1 1	3

added to the first one from the right, etc.) we have a string of length $N \times M$ in binary form with complexity K .

Strictly speaking we can estimate only the upper limit of K corresponding to a given algorithm. By algorithm (as it is defined above) we understand the computer program in PASCAL, along with the data file, describing the coordinates of the pixel of the spot. Namely the data file includes compressed information about the string of digits. The program is a sequence of commands performing reconstruction of the string and calculations of the corresponding lengths. Since at the analysis of various spots we will use the same program, the only change will be in the data files. Hence the complexity of the figure will be attributed to the file containing the information on the position of the pixels.

The code describing the spot works as follows. As an initial pixel we fix the upper left pixel of the spot and move clockwise along its boundary. Each step – a 'local step' – is a movement from a current pixel to the next one in above given direction. This procedure is rigorously defining the 'previous' and 'next' pixels. Two cases are possible. First, when the next pixel (or several pixels) after the initial one is in the same row: we write down the number of pixels in such 'horizontal step'. The second case is, when the next pixel is in vertical direction; then we perform the local steps in vertical direction ('vertical step') and record the number of corresponding pixels. Via a sequence of horizontal and vertical steps we, obviously, return to the initial pixel, thus defining the entire figure via a resulting data file.

Obviously, the length of the horizontal step cannot exceed the number of columns, i.e. N , while the vertical step cannot exceed M , requiring $\log_2 M$ and $\log_2 N$ bits of information, correspondingly. For the configurations we are interested in, the lengths of the horizontal and vertical steps, however, are much less than $\log_2 M$ and $\log_2 N$ and therefore we need a convenient code for defining the length of those steps. Our code is realized for $M = N = 256$; apparently for each value of M and N one has to choose the most efficient code.

Thus, after each step, either horizontal or vertical, certain amount of bits of information is stored. The first two bits will contain information on the following bits defining the length of the given step in a manner given in the following Table 1. The case when the first two bits are zero, denotes: if the following digit is zero than the length of the step is $l_s = 0$, and hence no digits of the same step do exist; if the next digit is 1, than 8 bits are following, thus defining the length of the step. If $l_s = 1$, than after the combination 01 the following digit will be either 0 or 1 depending whether the step is continued to the left or to the right with respect to the direction of the previous step. When $l_s = 2$ or 3, after the combination 01 the file records 0 in the first case, i.e. $l_s = 2$, and 1 in the second. When $l_s = 4, \dots, 7$, then after the combination 11 the file records 0 and 1, at the left and right steps, and after two digits in binary form of the step length $l_s = 3$. Finally, when $l_s > 8$, the combination 001 is recorded, followed by the 8 bits of the step length l_s in binary representation.

Table 2:

step length	bits
0, 1	3
2, 3	4
4-7	5
8	11

Thus, all possible values of the step length l_s (they are limited by $M = N = 256$) are taken into account and the amount of bits attributed to the length in the file depends on l_s in the manner shown in Table 2. The figure recorded in the data file via the described code can be recovered unambiguously without difficulties.

Obviously one cannot exclude the existence of a code compressing more densely the information on the pixelized spots, however even this codes appears to be rather efficient. Namely, the length of the program recovering the initial figure from the stored data file is 4908 bits, and it remains almost constant at the increase of N and M .

5 Hausdorff dimension

The association of local exponential instability and chaos with fractals is also well known (see e.g. [19]). Hence the idea to estimate the Hausdorff dimension of the spots is natural. We recall that, the Hausdorff dimension is defined as the limit

$$d = \lim_{\varepsilon \rightarrow 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)},$$

where $N(\varepsilon)$ are circles of radius ε covering at least one point of the set. By definition of Mandelbrot the set is fractal if Hausdorff dimension exceeds the topological dimension.

Our aim therefore should be to compute the d for the same studied objects-spots and look for the its behavior as compared with the complexity.

To compute the Hausdorff dimension we used the code *Fractal* by V.Nams [20]. . The main problem to be solved was the approximation of the boundary of the pixelized figure via a smooth curve, so that its Hausdorff dimension can be determined by the above mentioned code. The trivial consideration of the profile of the pixels, obviously would introduce artificial fractal properties to the spot as a result of instrumental nature of pixel sizes. We used the following procedure: the centers of three or more neighbour pixels were connected by a line and its distance h from the centers of the intermediate pixels has been calculated (it is obviously zero if the pixels are in one row). If h exceeds some chosen value, namely 0.5 of the size of the pixel, than the line was adopted as good approximation of the boundary curve of the pixels. Otherwise, the centers of the next pixels are involved, etc. The runs of test (trivial) figures with various values of h show the validity of this procedure.

6 Results

We now represent the results of computations of the complexity and Hausdorff dimensions by the described above algorithms for a sequence of computer-created spots. We start

from a most regular geometry - a circle, and after move to more 'complex' figures (Figure 1) as predicted for systems with strong mixing by the ergodic theory.

Figure 1.

Figure 2 plots the mutual variation of the complexity and Hausdorff dimensions.

Figure 2

By definition the data file has contribution in the value of complexity and, hence, the size of the spot will affect the results. We represent therefore also the dependence of the relative variation of the relative complexity $(K_i - K_1)/K_1$ and of the length of the data file on the size of the figures for the first two cases, including that of the circle (Figures 3a and 3b).

Figure 3.

7 Discussion

Thus we have represented a way of numerical computation of Kolmogorov complexity of a given configuration of pixels on a grid, thus imitating the anisotropy spots obtained during the CMB measurements. The importance of this descriptor is determined by the effect of geodesic mixing occurring in hyperbolic Universe. The latter among other observable consequences can lead to appearance of more 'complex' shapes of the anisotropies on the CMB sky maps.

It is known that the Kolmogorov complexity, i.e. the shortest program completely describing an object cannot be reached for typical objects. Similarly, the computer code that we used for the creation of the input data file containing the information on the spot and the estimation of the complexity though cannot be claimed to be the shortest one, nevertheless is appears to be efficient in our case. The main reason is the fact that we are interested in the relative complexity $K_i - K_1$ (or $(K_i - K_1)/K_1$) of two figures, given their interrelation due to the behavior of time correlations and hence with the KS-entropy of the geodesic flow [8], i.e. for photon beam motion on a negatively curved space. The relation of Kolmogorov complexity with KS-entropy which itself is determined by the curvature of the Friedmannian Universe [5], reveals the role of this new descriptor.

Our calculations showed the increase of the value of Kolmogorov complexity for more 'complex' spots, i.e. for the images corresponding to more later epochs of photon beam motion after the last scattering surface. The role of the size of the spots on the grid is also revealed. The complexity well correlates with the Hausdorff dimension of the spots.

Thus, we showed that an abstract quantity - Kolmogorov complexity, which is strictly speaking, non-calculable for typical systems, can be evaluated for CMB digitized maps; though more efficient codes than the one described in this paper, can be developed in future as well.

The next step will be to apply such codes to CMB real sky maps as it was done for the elongation parameter in [7] using the COBE-DMR data. We believe that this technique can be especially valuable for the analysis of the data to be obtained by forthcoming space and ground based experiments.

The valuable discussion with W.Zurek and the use of the program by V.O.Nams for calculation of the fractal dimension is greatly acknowledged.

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Figure captions.

Figure 1. The sequence of figures given by pixels on a grid 200 x 200 with the calculated values of (a) complexity K , (b) length of the data file D , and (c) Hausdorff dimension d_H .

Figure 2. The dependence of the Hausdorff dimension on the complexity for the same set of figures.

Figure 3. The dependence of the relative complexity $K_i - K_1/K_1$ (dashed line), where K_1 is the complexity of the circle, and the relative length of the data file (solid line) on the size of the corresponding figure. Figures (a) and (b) correspond to the first (circle) and the second image in Figure 1.

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